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Separability criterion induced from cross norm is not equivalent to positive partial transpose

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Abstract
Recently, a computational criterion of separability induced by the greatest cross norm was proposed by Rudolph (2002 *Preprint* quant-ph/0202121). There, Rudolph conjectured that the new criterion is neither weaker nor stronger than the positive partial-transpose criterion for separability. We show that there exists a counterexample to the this claim, that is, the proposed criterion is not equivalent to the positive partial-transpose criterion.

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1. Introduction

Entanglement as one of the most non-classical features of quantum mechanics has attracted much attention in the past decade. Although the non-local character of quantum mechanics was singled out many decades ago [1, 2], it has recently received considerable attention in connection with the theory of quantum information [3–6]. Entanglement usually arises from quantum correlations between separated subsystems which cannot be created by local actions on each subsystem. By definition, a bipartite mixed state ρ is said to be separable if it can be expressed as

$$\rho = \sum_i w_i \rho_i \otimes \sigma_i \qquad w_i \geqslant 0 \qquad \sum_i w_i = 1 \tag{1.1}$$

where ρ_i and σ_i denote density matrices of subsystems 1 and 2, respectively. Otherwise the state is entangled.

The central tasks of quantum information theory are to characterize and quantify entangled states. The first attempt at characterization of entangled states was made by Peres and Horodecki *et al* [7, 8]. It was shown that a necessary condition for separability of a bipartite system is that its partial transpose must be positive. Horodecki *et al* showed that this condition is sufficient for separability of composite systems only for dimensions $2 \otimes 2$ and $2 \otimes 3$.

A new criterion for separability and also an entanglement measure for bipartite systems based on the greatest cross norm are introduced by Rudolph [9–11]. In an interesting paper [11], he obtained the values of the greatest cross norm for some states such as Werner and isotropic states. In [11], Rudolph also introduced a computational criterion for separability of mixed states induced by the greatest cross norm and he could obtain the separability conditions for some states such as Werner states, isotropic states and 2-qubit Bell diagonal states. He showed that the new criterion completely characterizes the separability properties of pure states, Bell decomposable states and isotropic states in arbitrary dimension. He conjectured that the new criterion is neither weaker nor stronger than the positive partial transpose (PPT) criterion introduced by Peres and Horodecki *et al* in [7, 8].

In this paper, we introduce Bell decomposable states of $2 \otimes 3$ systems and show that there is a state in this category that is entangled in the sense of the PPT criterion but separable in the sense of the new criterion introduced in [11], that is the new criterion is not equivalent to the PPT criterion.

The paper is organized as follows. In section 2, we briefly review the greatest cross-norm criterion for separability of bipartite systems. The computable criterion induced by the greatest cross norm is also reviewed. Bell decomposable states in $2 \otimes 3$ systems are introduced in section 3 and the PPT conditions for separability of these states are also obtained. Finally, we show that there exists a state that is entangled in the sense of the PPT criterion but satisfies the new criterion for separability proposed by Rudolph.

2. Trace class-norm criterion for separability and associated induced separability criterion

In this section, we briefly review the greatest cross norm for separability of bipartite systems introduced by Rudolph in [9] and also the induced criterion introduced in [11].

By definition, the trace class norm of an operator A is defined by $\|A\|_1 = \mathrm{Tr}(\sqrt{A^\dagger A})$, and the operator A is said to be trace class operator if its trace class norm $\|A\|_1$ exists [12]. Let us consider Hilbert spaces H_1 and H_2 associated with particles 1 and 2, respectively. One can show that spaces $T(H_1)$ and $T(H_2)$ of trace class operators on H_1 and H_2 are Banach spaces once they are equipped with the trace-class norms $\|\cdot\|_1^{(1)}$ and $\|\cdot\|_1^{(2)}$, respectively. The algebraic tensor product $T(H_1) \otimes_{\mathrm{alg}} T(H_2)$ of $T(H_1)$ and $T(H_2)$ is defined as the set of all finite sums $\sum_{i=1}^n u_i \otimes v_i$, where $u_i \in T(H_1)$ and $v_i \in T(H_2)$ for all i.

A cross norm on $T(H_1) \otimes_{\mathrm{alg}} T(H_2)$ is defined by (see [9, 11] and references therein)

$$\|t\|_\gamma := \inf \left\{ \sum_{i=1}^n \|u_i\|_1 \|v_i\|_1 \,\middle|\, t = \sum_{i=1}^n u_i \otimes v_i \right\} \tag{2.2}$$

where $t \in T(H_1) \otimes_{\mathrm{alg}} T(H_2)$ and the infimum is taken over all finite decompositions of t into elementary tensors. By definition, a norm on $T(H_1) \otimes_{\mathrm{alg}} T(H_2)$ is a subcross norm if $\|t_1 \otimes t_2\| \leqslant \|t_1\|_1 \|t_2\|_1$ for all $t_1 \in T(H_1)$ and $t_2 \in T(H_2)$ and it is a cross norm if the inequality can be replaced by an equality for all $t_1 \in T(H_1)$ and $t_2 \in T(H_2)$. It can be shown that the norm $\|\cdot\|_\gamma$ majorizes any subcross norm on $T(H_1) \otimes_{\mathrm{alg}} T(H_2)$, so it is called greatest cross norm [13].

The greatest cross-norm criterion proposed by Rudolph is as follows [9, 11]. Let H_1 and H_2 be finite-dimensional Hilbert spaces and ρ be a density operator on $H_1 \otimes H_2$. The density matrix ρ is separable if and only if $\|\rho\|_\gamma = 1$. Rudolph in [11] determines the greatest cross norm for some states such as Werner and isotropic states. In addition, in the second part of [11], Rudolph introduced a new necessary separability criterion for bipartite systems induced by the greatest cross norm on the Hilbert–Schmidt space.

In the Hilbert–Schmidt space $HS(H)$, the operators of the Hilbert space H are regarded as vectors. This space is equipped with the Hilbert–Schmidt inner product defined by $\langle T|T'\rangle_{HS} = \mathrm{Tr}(T^{\dagger}T')$, where T and T' are two operators acting on space H.

Consider two finite-dimensional Hilbert spaces $K_1 \simeq C^n$ and $K_2 \simeq C^m$. It is shown in [11] that there is a one-to-one correspondence between states $|\psi\rangle \in K_1 \otimes K_2$ and Hilbert–Schmidt operators $A : K_2 \to K_1$ in such a way that if $|\psi\rangle = \sum_{ij} c_{ij}|a_i\rangle \otimes |b_j\rangle$ be a decomposition of $|\psi\rangle$ in terms of orthonormal bases $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ of K_1 and K_2, respectively, then $A(|\psi\rangle) = \sum_{ij} c_{ij}|a_i\rangle\langle b_j^*|$. Conversely, if $A = \sum_{ij} c_{ij}|a_i\rangle\langle b_j|$ is a decomposition of A in terms of orthonormal bases $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ of K_1 and K_2, respectively, then $|\psi_A\rangle = \sum_{ij} c_{ij}|a_i\rangle \otimes |b_j^*\rangle$. Rudolph also showed that the above one-to-one correspondence is isometric, that is, $\langle A(\psi_1)|A(\psi_2)\rangle_{HS} = \langle\psi_1|\psi_2\rangle$ and $\langle\psi_A|\psi_B\rangle = \langle A|B\rangle_{HS}$ [11].

Now consider the case that $K_1 = HS(H_1) \simeq C^n$ and $K_2 = HS(H_2) \simeq C^m$ are the spaces of Hilbert–Schmidt operators on Hilbert spaces H_1 and H_2, respectively. Analogous to the above correspondence, there exists a one-to-one correspondence between Hilbert–Schmidt operators $T \in HS(H_1 \otimes H_2)$ (acting on $H_1 \otimes H_2$) and Hilbert–Schmidt operators $\mathcal{U}(T) : HS(H_2) \to HS(H_1)$. Without loss of generality assume that $H_1 = H_2$. Every state $T \in HS(H \otimes H)$ in the Hilbert–Schmidt space can be written as [11]

$$T = \sum_i \lambda_i E_i \otimes F_i \tag{2.3}$$

where $\{\lambda_i\}_i$ are non-negative real numbers and $\{E_i\}_i$ and $\{F_i\}_i$ are orthonormal bases of Hilbert–Schmidt spaces of particles 1 and 2, respectively [11]. It follows that the trace-class norm of $\mathcal{U}(T)$ denoted by $\mathcal{T}(\mathcal{U}(T))$ is equal to $\mathcal{T}(\mathcal{U}(T)) = \sum_i \lambda_i$.

Rudolph in [11] proposed its new criterion for separability in a proposition which is quoted below:

Proposition 1 [11]**.** *Let H be a finite-dimensional Hilbert space and $\rho \in T(H \otimes H)$ be a density operator. If ρ is separable then*

$$\mathcal{T}(\mathcal{U}(\rho)) \leqslant 1. \tag{2.4}$$

Based on the above criterion, Rudolph obtained separability conditions of some states such as Werner states, isotropic states and 2-qubit Bell diagonal states. He conjectured that the new criterion is neither weaker nor stronger than the Peres–Horodecki PPT criterion for separability. In the following section, we present a state that violates the positive partial-transpose criterion but satisfies the separability criterion given in equation (2.4).

3. Bell decomposable states

In this section, we review Bell decomposable states of $2 \otimes 3$ quantum systems. A Bell decomposable density matrix acting on $2 \otimes 3$ Hilbert space can be defined by

$$\rho = \sum_{i=1}^{6} p_i |\psi_i\rangle\langle\psi_i| \quad 0 \leqslant p_i \leqslant 1 \qquad \sum_{i=1}^{6} p_i = 1 \tag{3.5}$$

where $|\psi_i\rangle$ are Bell states in $H^2 \otimes H^3 \cong H^6$ Hilbert space, defined by

$$\begin{aligned} |\psi_1\rangle &= \tfrac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \qquad & |\psi_2\rangle &= \tfrac{1}{\sqrt{2}}(|11\rangle - |22\rangle) \\ |\psi_3\rangle &= \tfrac{1}{\sqrt{2}}(|12\rangle + |23\rangle) \qquad & |\psi_4\rangle &= \tfrac{1}{\sqrt{2}}(|12\rangle - |23\rangle) \\ |\psi_5\rangle &= \tfrac{1}{\sqrt{2}}(|13\rangle + |21\rangle) \qquad & |\psi_6\rangle &= \tfrac{1}{\sqrt{2}}(|13\rangle - |21\rangle). \end{aligned} \tag{3.6}$$

It is quite easy to see that the above states are orthogonal and thus span the Hilbert space of $2 \otimes 3$ systems.

A necessary condition for separability is presented by Peres [7]. He showed that the matrix obtained from the partial transpose of a separable state must be positive. Horodecki *et al* in [8] have shown that the Peres criterion provides a sufficient condition for separability only for the composite quantum systems of dimensions $2 \otimes 2$ and $2 \otimes 3$. This implies that the state given in equation (3.5) is separable if and only if the following inequalities satisfy:

$$(p_1 + p_2)(p_3 + p_4) \geqslant (p_5 - p_6)^2 \tag{3.7}$$

$$(p_3 + p_4)(p_5 + p_6) \geqslant (p_1 - p_2)^2 \tag{3.8}$$

$$(p_5 + p_6)(p_1 + p_2) \geqslant (p_3 - p_4)^2. \tag{3.9}$$

Now let us expand ρ given in equation (3.5) in terms of canonical bases $|i\rangle \otimes |\alpha\rangle$ (for $i = 1, 2$ and $\alpha = 1, 2, 3$) as

$$\begin{aligned} \rho = \tfrac{1}{2}((p_1 + p_2)|11\rangle\langle 11| &+ (p_1 - p_2)|11\rangle\langle 22| + (p_1 - p_2)|22\rangle\langle 11| \\ &+ (p_1 + p_2)|22\rangle\langle 22| + (p_3 + p_4)|12\rangle\langle 12| + (p_3 - p_4)|12\rangle\langle 23| \\ &+ (p_3 - p_4)|23\rangle\langle 12| + (p_3 + p_4)|23\rangle\langle 23| + (p_5 + p_6)|13\rangle\langle 13| \\ &+ (p_5 - p_6)|13\rangle\langle 21| + (p_5 - p_6)|21\rangle\langle 13| + (p_5 + p_6)|21\rangle\langle 21|). \end{aligned} \tag{3.10}$$

It can be seen that by defining $E_{ij} = |i\rangle\langle j|$ and $F_{\alpha\beta} = |\alpha\rangle\langle\beta|$ as orthonormal bases of $HS(H_1)$ and $HS(H_2)$, respectively, equation (3.10) can be written as $\rho = \sum_{ij,\alpha\beta} C_{ij,\alpha\beta} E_{ij} \otimes F_{\alpha\beta}$. Using this notation, we can evaluate the corresponding Hilbert–Schmidt operator $\mathcal{U}(\rho)$ as

$$\begin{aligned} \mathcal{U}(\rho) = \tfrac{1}{2}((p_1 + p_2)|E_{11}\rangle\langle E_{11}| &+ (p_1 - p_2)|E_{12}\rangle\langle E_{12}| + (p_1 - p_2)|E_{21}\rangle\langle E_{21}| \\ &+ (p_1 + p_2)|E_{22}\rangle\langle E_{22}| + (p_3 + p_4)|E_{11}\rangle\langle E_{22}| + (p_3 - p_4)|E_{12}\rangle\langle E_{23}| \\ &+ (p_3 - p_4)|E_{21}\rangle\langle E_{32}| + (p_3 + p_4)|E_{22}\rangle\langle E_{33}| + (p_5 + p_6)|E_{11}\rangle\langle E_{33}| \\ &+ (p_5 - p_6)|E_{12}\rangle\langle E_{31}| + (p_5 - p_6)|E_{21}\rangle\langle E_{13}| + (p_5 + p_6)|E_{22}\rangle\langle E_{11}|) \end{aligned} \tag{3.11}$$

where $|E_{ij}\rangle \equiv |i\rangle\langle j|$ and $\langle E_{\alpha\beta}| \equiv |\alpha\rangle\langle\beta|$ are used to denote E_{ij} and $F_{\alpha\beta}$, respectively. Now we can easily evaluate the eigenvalues of the 4×4 matrix $\mathcal{U}(\rho)\mathcal{U}^\dagger(\rho)$ which yields

$$\mu_1 = \mu_2 = A \qquad \mu_3 = B + C \qquad \mu_4 = B - C \tag{3.12}$$

where

$$\begin{aligned} A &= \tfrac{1}{4}((p_1 - p_2)^2 + (p_3 - p_4)^2 + (p_5 - p_6)^2) \\ B &= \tfrac{1}{4}((p_1 + p_2)^2 + (p_3 + p_4)^2 + (p_5 + p_6)^2) \\ C &= \tfrac{1}{4}((p_1 + p_2)(p_3 + p_4) + (p_3 + p_4)(p_5 + p_6) + (p_5 + p_6)(p_1 + p_2)). \end{aligned} \tag{3.13}$$

It is easy to see that all eigenvalues are non-negative. Now by evaluating the trace class norm $\mathcal{T}(\mathcal{U}(\rho)) = \mathrm{Tr}(\sqrt{\mathcal{U}(\rho)\mathcal{U}(\rho)^\dagger}) = \sum_i \sqrt{\mu_i}$, we can easily determine the separability criterion given in equation (2.4) as

$$\mathcal{T}(\mathcal{U}(\rho)) = 2\sqrt{A} + \sqrt{B + C} + \sqrt{B - C} \leqslant 1. \tag{3.14}$$

In the rest of this section, we shall present a counterexample to the claim that the criterion given in equation (2.4) is neither weaker nor stronger than the PPT criterion for separability. Let us consider a Bell decomposable state given by

$$p_1 = 0.3 \qquad p_2 = 0 \qquad p_3 = 0.2 \qquad p_4 = 0.1 \qquad p_5 = 0.4 \qquad p_6 = 0. \tag{3.15}$$

It is quite easy to see that the state given by equation (3.15) violates the PPT criterion given in equation (3.7), so it is an entangled state. On the other hand, it is separable in the sense of criterion given in equation (2.4). It follows that any $2 \otimes 3$ state that satisfies the PPT criterion is separable thus also satisfies the necessary criterion given in equation (2.4). Therefore, there exists an entangled state that does not violate the new criterion. This means that as far as $2 \otimes 3$ systems are concerned, the new criterion given in equation (2.4) is weaker than the PPT criterion. On the other hand, Rudolph in [11] presented an entangled state of a $3 \otimes 3$ system (see also [14]) that violates the new criterion but satisfies the PPT criterion. This implies that, in general, the new criterion induced by the greatest cross norm is not equivalent to the PPT criterion for separability.

4. Conclusion

We have provided a counterexample of $2 \otimes 3$ systems to show that the newly proposed criterion of separability (induced by the greatest cross norm) proposed by Rudolph is not equivalent to the positive partial-transpose criterion; therefore, we have still a long way to go to solve the long-standing separability criterion in mixed quantum states.

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